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Kolmogorov Exponents for Near-Incompressible Turbulence from Perturbative Quantum Field Theory

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I derive the Kolmogorov exponents for the energy spectrum of freely-decaying, fully-developed, near-incompressible turbulence, using the methods of perturbative quantum field theory. In contrast to the approach involving Gaussian random forces, the leading-order result is determined uniquely through self-consistency. At the first order in ε, I find a unique and nontrivial, IR (infrared) stable fixed-line. I show that the upper critical dimension of this system is 6, and $E(k) \approx k^{-2}$ in 3 dimensions and $E(k) \approx k^{-3}$ in 2 dimensions along this nontrivial fixed-line (at the one-loop level).

KEY WORDS: Dimensional regularization; minimal subtraction; Kolmogorov inertial range; renormalization; beta function; near-incompressible turbulence; fixed points; scaling behavior; leading scaling exponents; counterterm.

1. INTRODUCTION

An outstanding problem in many-body physics is to find an analytical derivation of the scaling behavior of fully-developed turbulence. The universal scaling exponents in the inertial range have been checked experimentally for a large variety of turbulent flows. The Kolmogorov law has been confirmed experimentally in fluid and gas shear flow, in turbulence behind a grid, and in atmospheric boundary layers.[1] It is commonly accepted in the fluid community that, in the Kolmogorov inertial range, $E(k) \approx k^{-5/3-B}$ (with $B \approx 0.17$ the so-called intermittency

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exponent[2]) in 3 dimensions and $E(k) \approx k^{-3}$ in 2 dimensions.[3,4] Here, $E(k)$ is defined by $\langle v(x,t)\, v(x,t) \rangle \equiv \int_0^\infty E(k)\, dk$. It is implicitly assumed that the state is quasistationary in the region of fully-developed turbulence, and therefore the equal-time correlation functions are approximately time independent.

After the initial success of the renormalization-group (RG) technique in explaining the scaling behavior of the dynamics of the second-order phase transition, many authors attempted to apply the same methods to derive the scaling exponents in the fully-developed turbulence.[5–7] Unfortunately, within the limits set by the renormalizability of the theory, the derived exponent *strongly* depends on the scaling form of the assumed force–force correlation function.[5] This seems to contradict the observed universality of the Kolmogorov exponent: the scaling exponent of the equal-time velocity–velocity correlation function in the fully-developed turbulence region seems to be identical for widely different experimental situations.[1]

Physically, one can understand the reason for the failure of the above approach as follows. In the critical dynamics, we study the equations of motion of the order parameters when the system is slightly out of equilibrium. By the fluctuation-dissipation theorem, the amplitude of the thermal random force has to be related to the dissipative part of the system in order for the system to reach thermal equilibrium in the long-time limit. Mathematically, this manifests itself as the fact that the operator dimensionality of the order-parameter field will depend on the assumed scaling form of the "thermal" force. Physically, what has been done is to put the external thermal force and the dissipative part of the equation on the same footing (as required by the fluctuation-dissipation theorem). One then treats the nonlinear part of the equation as a perturbation within the framework of the ε expansion. The form of the correlation function of the thermal force plays an important role here, because it determines the "way" the system is brought back to thermal equilibrium.

However, the physics of fully-developed turbulence is believed to be dominated by the nonlinear part in comparison to the viscous-dissipative part of the equation. There is no physical reason to introduce an external agent into the problem. Any randomness in the flow field is due to the intrinsic instability of the equations of motion, not an external random force field. What has been done in the literature is to study the Gaussian-randomly forced Navier–Stokes equation. As explained in the previous paragraph, this approach puts the viscous-dissipative part and the arbitrarily-chosen random force on an equal footing. *Together, they determine the zeroth-order propagator.* Since the form of the *zeroth*-order propagator is determined by the form of the external force one chooses, one should not be surprised by the fact that "what comes out depends on what goes in."

In this article I shall present an alternative approach, using perturbative quantum field theory. I shall address the question of the scaling behavior of freely-decaying, fully-developed, and near-incompressible turbulence, without introducing an external arbitrarily-chosen random force. (This may also be construed as the statement that the renormalized force amplitude has been properly tuned to zero.[8]) At the one-loop level, I shall present evidence that the upper critical dimension of this system is 6, via the existence of a unique, infrared (IR) stable, and nontrivial fixed-line. I shall then show that, along this nontrivial fixed-line, $E(k) \approx k^{-2}$ in 3 dimensions and $E(k) \approx k^{-3}$ in 2 dimensions. Hence, without any *ad hoc* assumptions, I have calculated two independent exponents to within 10% of the best available experimental measurements.

2. FORMULATION OF THE THEORY

It is a common belief that the apparent chaotic and intermittent flow patterns of the fully-developed turbulence result from the nonlinear interaction between different Fourier modes of the velocity field. This nonlinear interaction comes from the nonlinear terms in the equations of motion. Hence, we will focus our study on the effects of the nonlinear interaction in the equations of motion. The simplest model is the classical one-component fluid (i.e., a fluid with one molecular species) without long-range interaction. We will neglect complications due to temperature gradients, chemical potential differentials, external force fields, and quantum effects. The state of the fluid is then uniquely determined by the velocity field, the density field, and the pressure field. The equations of motion of the fluid are consequences of the momentum and mass conservation laws[9]:

$$\frac{\partial}{\partial t}(\rho v_\alpha) = -\frac{\partial \Pi_{\alpha\beta}}{\partial x_\beta} \tag{2.1}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2.2}$$

where ρ is the density of the fluid; $\mathbf{v}$ is the velocity field; and $\Pi_{\alpha\beta}$ is the momentum flux density tensor. Following Landau and Lifshitz,[9] we write the momentum flux density tensor for a viscous fluid in the form

$$\Pi_{\alpha\beta} = P\delta_{\alpha\beta} + \rho v_\alpha v_\beta - \sigma'_{\alpha\beta} \tag{2.3}$$

where P is the pressure and $\sigma'_{\alpha\beta}$ is the viscosity stress tensor.

The most general form of $\sigma'_{\alpha\beta}$ can be established as follows.[9] We know that the internal friction occurs in a fluid only when there are velocity differentials between different parts of the fluid. Since there is no long-range force in the fluid, the stress tensor must be *local.* Furthermore, all the basic fields are locally "coarse-grained" quantities. The equations of motion must contain only terms that are analytic in the basic fields. *This is the fundamental assumption of locality and analyticity.* The lowest-order terms (in gradients and in velocity fields) of $\sigma'_{\alpha\beta}$ are of the following form:

$$\sigma'_{\alpha\beta} \approx \eta \left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} - \frac{2}{3} \delta_{\alpha\beta} \frac{\partial v_\gamma}{\partial x_\gamma} \right) + \zeta \delta_{\alpha\beta} \frac{\partial v_\gamma}{\partial x_\gamma} \tag{2.4}$$

where $\eta > 0$ and $\zeta > 0$ are coefficients of viscosity.

By the same reasoning, the coefficients of viscosity, namely η and ζ, are local and analytical functions of ρ and $\mathbf{v}$. As will become clear later, it is sufficient to keep just the constant pieces in this expansion. Higher-order terms (in gradients, ρ, and $\mathbf{v}$) will turn out to be IR irrelevant, as far as the scaling exponents are concerned. Then Eq. (2.1) becomes

$$\rho \left(\frac{\partial v_\alpha}{\partial t} + v_\beta \partial_\beta v_\alpha \right) = -\partial_\alpha P + \bar{\eta} \partial^2 v_\alpha + \bar{\zeta}\, \partial_\alpha \partial_\beta v_\beta \tag{2.5}$$

where $\bar{\eta}$ and $\bar{\zeta}$ are constants.

We will focus our study on the case where the fluid is near-incompressible. This is indeed the situation where all experimental measurements are carried out. We write $\rho = \rho_0 + \delta\rho$ and denote $\hat{\rho} = \delta\rho/\rho_0$. Here ρ_0 is a constant throughout the space-time. Then Eqs. (2.1) and (2.2) become

$$\frac{\partial v_\alpha}{\partial t} + v_\beta \partial_\beta v_\alpha + \hat{\rho} \left(\frac{\partial v_\alpha}{\partial t} + v_\beta \partial_\beta v_\alpha \right) = -\frac{\partial_\alpha P}{\rho_0} + \frac{\bar{\eta}}{\rho_0} \partial^2 v_\alpha + \frac{\bar{\zeta}}{\rho_0} \partial_\alpha \partial_\beta v_\beta \tag{2.6}$$

$$\frac{\partial \hat{\rho}}{\partial t} + \partial_\alpha v_\alpha + \partial_\alpha (\hat{\rho} v_\alpha) = 0 \tag{2.7}$$

We have to close Eqs. (2.6) and (2.7) by expressing P as a function of ρ and $\mathbf{v}$. Again, the guiding principle in this expansion is the requirement of locality and analyticity. The fact that P is a Galilean scalar implies that the lowest-order terms in the expansion look like

$$P \approx c_1 \rho + c_2 \partial_\beta v_\beta + \cdots \tag{2.8}$$

where c_1 and c_2 are constants. The strategy will be to show, later, that all higher-order terms in this expansion are IR irrelevant in the RG sense.

Let us denote $\nu_0 \equiv \bar{\eta}/\rho_0$ and $b_0 \equiv (\bar{\zeta} - c_2)/\bar{\eta}$. Obviously, we must have $\nu_0 \geqslant 0$ and $b_0 \geqslant 0$, because they are the coefficients of viscosity. Hence, the set of equations that we have to study is

$$\mathrm{NS}[v_\alpha, \hat{\rho}] \equiv \frac{1}{\nu_0} \frac{\partial v_\alpha}{\partial t} + \frac{v_\beta \partial_\beta v_\alpha}{\nu_0} + \frac{\hat{\rho}}{\nu_0} \left(\frac{\partial v_\alpha}{\partial t} + v_\beta \partial_\beta v_\alpha \right) + \frac{c_1}{\nu_0} \partial_\alpha \hat{\rho} - \partial^2 v_\alpha - b_0 \partial_\alpha \partial_\beta v_\beta = 0 \tag{2.9}$$

$$\mathrm{CON}[v, \hat{\rho}] \equiv \frac{\partial \hat{\rho}}{\partial t} + \partial_\alpha v_\alpha + \partial_\alpha(\hat{\rho} v_\alpha) = 0 \tag{2.10}$$

The symbol NS reminds us that they are the Navier–Stokes equations, and CON reminds us that it is the equation of continuity (from the mass conservation law).

3. RENORMALIZATION GROUP CALCULATION

We will average over all solutions of $\mathrm{NS}[v_\alpha, \hat{\rho}] = 0$ *and* $\mathrm{CON}[v, \hat{\rho}] = 0$ *with equal weight and impose the condition that* $\langle v_\alpha \rangle = 0$. This theoretical formulation is modeled after the freely-decaying turbulence experiments.[3,10] In the freely-decaying grid-turbulence experiment, one shakes the grid vigorously to ensure that the fluid is in the fully-developed turbulence region. At $t = 0$, the grid oscillation is turned off suddenly. One observes that within a short time, the flow pattern becomes rather homogeneous. One makes repeated measurements and computes the time average to obtain the "statistical properties" of the freely-decaying turbulence. For the two-point correlation function, one is measuring the following quantity:

$$\frac{1}{N} \sum_{n=0}^{N} v_\alpha(\mathbf{r}, T_1 + n\Delta t)\, v_\beta(\mathbf{r}', T_1 + n\Delta t) \tag{3.1}$$

$$N = \frac{T_2 - T_1}{\Delta t} \tag{3.2}$$

where T_1 is the beginning and T_2 is the end of the time interval when the measurements are carried out. Δt is the time span between two successive measurements. One can also interpret the above as the *ensemble average with equal weight*: $v_\alpha(\mathbf{r}, T_1 + i\Delta t)$ is the αth-component velocity field configuration after time T_1 from the ith "initial" condition, where the ith "initial" condition is the field configuration at time $t = i\Delta t$. One also notices that there is no mean flow in this experiment: $\langle v_\alpha \rangle = 0$.

One might object that the ith and $(i+1)$th "initial" conditions are related through the equations of motion. The crucial point to bear in mind, however, is that (from numerous experiments) the leading scaling exponent of the velocity–velocity correlation function in the fully-developed turbulence seems to be independent of the initial preparations. Hence, we can equate the time average with the ensemble average. Since the averaging procedure is linear, we should be able to compute the correct leading scaling exponent by averaging over all solutions of $\mathrm{NS}[v_\alpha, \hat{\rho}] = 0$ and $\mathrm{CON}[v, \hat{\rho}] = 0$ with equal weight and look for the stationary state with $\langle v_\alpha \rangle = 0$. In a sense, *we are assuming universality from the experimental evidence.*

The generating functional of all physical correlation functions can be written symbolically as

$$\begin{aligned} Z[l, J] &\equiv \sum_{v_{\mathrm{sol}},\, \hat{\rho}_{\mathrm{sol}}} \exp\{l v_{\mathrm{sol}} + J \hat{\rho}_{\mathrm{sol}}\} \\ &\equiv \sum_{v_{\mathrm{sol}},\, \hat{\rho}_{\mathrm{sol}}} \int \mathscr{D}v\, \mathscr{D}\hat{\rho}\, \delta(v - v_{\mathrm{sol}})\, \delta(\hat{\rho} - \hat{\rho}_{\mathrm{sol}}) \exp\{l v + J \hat{\rho}\} \\ &\equiv \int \mathscr{D}v\, \mathscr{D}\hat{\rho}\, \delta(\nabla^{2n}\mathrm{NS})\, \delta(\mathrm{CON})\, J[v, \hat{\rho}] \exp\{l v + J\hat{\rho}\} \end{aligned} \tag{3.3}$$

where v_{sol} and $\hat{\rho}_{\mathrm{sol}}$ denote solutions of $\mathrm{NS}[v_\alpha, \hat{\rho}] = 0$ and $\mathrm{CON}[v, \hat{\rho}] = 0$. Here $J[v, \hat{\rho}]$ is the functional Jacobian coming from the argument of the δ function[11]:

$$\begin{aligned} &\delta(\nabla^{2n}\mathrm{NS})\, \delta(\mathrm{CON}) \\ &\quad \equiv \frac{\delta(\mathrm{NS})\, \delta(\mathrm{CON})}{\det |\nabla^{2n}|} = \sum_{v_{\mathrm{sol}},\, \hat{\rho}_{\mathrm{sol}}} \frac{\delta(v - v_{\mathrm{sol}})\, \delta(\hat{\rho} - \hat{\rho}_{\mathrm{sol}})}{J[v, \hat{\rho}] \det |\nabla^{2n}|} \end{aligned} \tag{3.4}$$

$$J[v, \hat{\rho}] \propto \exp\left\{ -\frac{1}{2a^d} \int_a d^d\mathbf{x}\, dt \left[\frac{\partial F_\alpha}{\partial v_\alpha} + \partial_\alpha v_\alpha \right] \right\} \tag{3.5}$$

$$F_\alpha[v] \equiv \mathrm{NS}[v_\alpha, \hat{\rho}] - \frac{1}{v_0} \frac{\partial v_\alpha}{\partial t} \tag{3.6}$$

In Eq. (3.5), a denotes the underlying "lattice cutoff." The presence of the lattice cutoff will be explained shortly. Since we are going to use *dimensional regularization* throughout this article, we can drop contributions from $J[v, \hat{\rho}]$ entirely.[12] The reason is that the factor a^{-d} can be written as $(1/2\pi) \int^{\Lambda} d^d\mathbf{q}$, which vanishes if dimensionally regularized.[15] (Here, the presence of the upper momentum cutoff Λ accounts for the underlying lattice structure.) This is similar to the disposal of the interactions induced by the measure in the nonlinear sigma model regularized by the dimensional regularization.

Let us now explain the reason for considering solutions of $\nabla^{2n}\mathrm{NS}=0$, instead of $\mathrm{NS}=0$. We know from experimental observation that there are "eddies within eddies" and intermittency effects in the fully-developed turbulence. This suggests that the appropriate focus of study might not be the velocity field itself, but *higher-order spatial derivatives* of the velocity field. Here, we propose the following ansatz: We should study the following equations: $\nabla^{2n}\mathrm{NS}[v_\alpha, \hat{\rho}]=0$ and $\mathrm{CON}[v, \hat{\rho}]=0$, where $n=0, 1, 2, \ldots$. However, one has to show that in considering $\nabla^{2n}\mathrm{NS}=0$, one does not introduce additional field configurations that are not solutions of the original equations of motion. One notices that, since we impose the spatial boundary condition that all fields vanish at spatial infinity, $\mathrm{NS}[v_\alpha, \hat{\rho}]=0$ is always true at the spatial infinity. Hence, $\nabla^{2n}\mathrm{NS}[v_\alpha, \hat{\rho}]=0$ has only one solution, namely $\mathrm{NS}[v_\alpha, \hat{\rho}]=0$ at all space-time points. The *unique* choice of n will become clear in Section 4.

One might ask: Why should we consider different choices of n if mathematically they have no effect when the problem are treated *exactly*? The point is that we cannot solve the problem exactly. The best one can do is to construct the theory perturbatively. Then, different choices of n produce different answers because the expansion points for the perturbation theories are different. One expects that proper physical consideration will guide us to the correct expansion point. Indeed, one of the major contributions of this article is to show how the requirement of locality, analyticity, and renormalizability will restrict the possible choices of n to a few finite numbers.

More specifically, the generating functional of all physical correlation functions in real-space is

$$
\begin{aligned}
Z[I(\mathbf{x}, t), J(\mathbf{x}, t)] \\
&\equiv \int [\mathscr{D}\hat{\rho}(\mathbf{x}, t)][\mathscr{D}\hat{\sigma}'(\mathbf{x}, t)] \prod_{\alpha=1}^{d} [\mathscr{D}v_\alpha(\mathbf{x}, t)][\mathscr{D}\phi'_\alpha(\mathbf{x}, t)] \\
&\quad \times \exp\left\{\int_a d^d\mathbf{x}\, dt [I_\alpha(\mathbf{x}, t)\, v_\alpha(\mathbf{x}, t) + J(\mathbf{x}, t)\, \hat{\rho}(\mathbf{x}, t)]\right\} \\
&\quad \times \exp\left\{-i\int_a d^d\mathbf{x}\, dt [\phi'_\alpha(\mathbf{x}, t)\, \nabla^{2n}\mathrm{NS}[v_\alpha, \hat{\rho}] + \hat{\sigma}'(\mathbf{x}, t)\, \mathrm{CON}[v, \hat{\rho}]]\right\}
\end{aligned}
\tag{3.7}
$$

In writing Eq. (3.7), we have used the Fourier functional-integral representation of the δ function:

$$\prod_{\alpha=1}^{d} \prod_{\mathbf{x},t} \delta(\nabla^{2n}\mathrm{NS}[v_\alpha, \hat{\rho}])\, \delta(\mathrm{CON}[v, \hat{\rho}])$$

$$\equiv \int [\mathscr{D}\hat{\rho}(\mathbf{x}, t)][\mathscr{D}\hat{\sigma}'(\mathbf{x}, t)] \prod_{\alpha=1}^{d} [\mathscr{D}v_\alpha(\mathbf{x}, t)][\mathscr{D}\phi'_\alpha(\mathbf{x}, t)]$$

$$\times \exp\left\{-i \int_a d^d\mathbf{x}\, dt [\phi'_\alpha(\mathbf{x}, t)\, \nabla^{2n}\mathrm{NS}[v_\alpha, \hat{\rho}] + \hat{\sigma}'(\mathbf{x}, t)\, \mathrm{CON}[v, \hat{\rho}]]\right\} \tag{3.8}$$

In Eqs. (3.5), (3.7), and (3.8), we notice that there is a lattice cutoff of the order a in the spatial integral. The reason for the underlying lattice cutoff is as follows. One knows that the Navier–Stokes equation is a "coarse-grained" equation of motion. The velocity field loses its meaning at the molecular scale. It is a locally averaged quantity of a "fluid element," which might consist of hundreds or thousands of molecules. Hence, *from the physical point of view*, there is an intrinsic underlying lattice cutoff, beneath which the hydrodynamic description of the fluid becomes invalid. On the other hand, one can also regard Eqs. (2.9) and (2.10) as a system of partial differential equations *per se*. Thus, one is not concerned about the lattice cutoff, because space and time are continuous. However, we are dealing with real physics here. This is the reason for the emphasis on the underlying lattice cutoff. This is exactly the same situation as in any other physical system studied in condensed matter physics.

It is easy to see that Eq. (3.7) is equivalent to the following:

$$Z[\mathbf{l}(\mathbf{x}, t), J(\mathbf{x}, t)]$$

$$\equiv \int [\mathscr{D}\hat{\rho}(\mathbf{x}, t)][\mathscr{D}\hat{\sigma}'(\mathbf{x}, t)] \prod_{\alpha=1}^{d} [\mathscr{D}v_\alpha(\mathbf{x}, t)][\mathscr{D}\phi_\alpha(\mathbf{x}, t)]$$

$$\times \exp\left\{\int_a d^d\mathbf{x}\, dt [l_\alpha(\mathbf{x}, t)\, v_\alpha(\mathbf{x}, t) + J(\mathbf{x}, t)\, \hat{\rho}(x, t)]\right\}$$

$$\times \exp\left\{-i \int_a d^d\mathbf{x}\, dt\, \hat{\sigma}'(\mathbf{x}, t)\, \mathrm{CON}[v, \hat{\rho}]\right\}$$

$$\times \exp\left\{-i \int_a d^d\mathbf{x}\, dt\, \phi_\alpha(\mathbf{x}, t)\, \nabla^{2n}\left[\mathrm{NS}[v_\alpha, \hat{\rho}] + \frac{e_1}{v_0} v_\alpha\, \mathrm{CON}[v, \hat{\rho}]\right]\right\} \tag{3.9}$$

with e_1 some arbitrary constant. Indeed, when one integrates out $\hat{\sigma}'$ and $\boldsymbol{\phi}$, the only configurations of $\mathbf{v}$ and $\hat{\rho}$ with nonzero contribution to the path integral are those satisfying CON $=0$ and NS $+ (e_1/v_0)\, v_\alpha$ CON $=0$. That is, they are solutions of CON $=0$ and NS $=0$. Finally, one concludes that Eq. (3.9) is equivalent to

$$
\begin{aligned}
&Z[\mathbf{I}(\mathbf{x}, t), J(\mathbf{x}, t)] \\
&\quad \equiv \int [\mathscr{D}\hat{\rho}(\mathbf{x}, t)][\mathscr{D}\hat{\sigma}(\mathbf{x}, t)] \prod_{\alpha=1}^{d} [\mathscr{D}v_\alpha(\mathbf{x}, t)][\mathscr{D}\phi_\alpha(\mathbf{x}, t)] \\
&\qquad \times \exp\left\{\int_a d^d\mathbf{x}\, dt [I_\alpha(\mathbf{x}, t)\, v_\alpha(\mathbf{x}, t) + J(\mathbf{x}, t)\, \hat{\rho}(\mathbf{x}, t)]\right\} \\
&\qquad \times \exp\left\{-i \int_a d^d\mathbf{x}\, dt\, \phi_\alpha(\mathbf{x}, t)\, \nabla^{2n}\left[\mathrm{NS}[v_\alpha, \hat{\rho}] + \frac{e_1}{v_0} v_\alpha \mathrm{CON}[v, \hat{\rho}]\right]\right\} \\
&\qquad \times \exp\left\{-i \int_a d^d\mathbf{x}\, dt\, \hat{\sigma}(\mathbf{x}, t)\left[\mathrm{CON}[v, \hat{\rho}] + e_2 \partial_\alpha \mathrm{NS}[v_\alpha, \hat{\rho}]\right.\right. \\
&\qquad \left.\left. + \frac{e_1 e_2}{v_0} \partial_\alpha(v_\alpha \mathrm{CON}[v, \hat{\rho}])\right]\right\} \\
&\qquad \times \exp\left\{\int_a d^d\mathbf{x}\, dt\, f_\alpha[v, \hat{\rho}]\, \mathrm{NS}[v_\alpha, \hat{\rho}]\right\} \\
&\qquad \times \exp\left\{\int_a d^d\mathbf{x}\, dt\, g[v, \hat{\rho}]\, \mathrm{CON}[v, \hat{\rho}]\right\}
\end{aligned}
\tag{3.10}
$$

where e_2 is some arbitrary constant and $f_\alpha[v, \hat{\rho}]$ and $g[v, \hat{\rho}]$ are some local functions of $\mathbf{v}$ and $\hat{\rho}$.

The reason for introducing these "extra operators" in the action is to provide enough counterterms in order to make the theory renormalizable when we introduce the source terms for ϕ and $\hat{\sigma}$ fields. The renormalized coupling constants of these "extra operators" must be zero, because the renormalized equations of motion should have the same form as in Eqs. (2.9) and (2.10). The situation here is similar to the study of the tricritical point,[13] where the renormalized coupling constant of S^4 is zero. The renormalized nonlinear term now is S^6. However, we still need the counterterm for S^4 in the bare level, because S^4 will be generated under the renormalization. From now on, we will not write these "extra operators" explicitly. However, we have to remember that they are in fact present in the counterterm action in order to make the theory renormalizable. One such example is

$$\phi_\alpha \nabla^{2n}\left[\frac{e_1}{v_0} v_\alpha \partial_\beta v_\beta\right] \tag{3.11}$$

which is missing in

$$\phi_\alpha \nabla^{2n} \mathrm{NS}[v_\alpha, \hat{\rho}] \tag{3.12}$$

but will in general be generated under renormalization. The "extra operator" in Eq. (3.11) comes from the factor $(e_1/v_0)\, v_\alpha$ CON in Eq. (3.10). It differs from the basic nonlinear term $v_\beta \partial_\beta v_\alpha$ only in the order of the vector indices.

Let us introduce the sources h_α and I for the auxiliary fields ϕ_α and $\hat{\sigma}$. Since we are interested in translationally-invariant and stationary states, we will work in the Fourier space:

$$v_\alpha(\mathbf{x}, t) \equiv \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{dw}{2\pi}\, v_\alpha(\mathbf{k}, w) \exp\{i\mathbf{k}\cdot\mathbf{x} - iwt\} \tag{3.13}$$

$$v_\alpha(\mathbf{k}, w) \equiv \int d^d\mathbf{x}\, dt\, v_\alpha(\mathbf{x}, t) \exp\{-i\mathbf{k}\cdot\mathbf{x} + iwt\} \tag{3.14}$$

The full generating functional now becomes ("extra operators" are not explicitly expressed in the following formulas):

$$\begin{aligned} Z \equiv \int & [\mathscr{D}\hat{\rho}][\mathscr{D}\hat{\sigma}][\mathscr{D}v_\alpha][\mathscr{D}\phi_\alpha] \exp\{\langle\!\langle v_\alpha, l_\alpha \rangle\!\rangle + \langle\!\langle \phi_\alpha, h_\alpha \rangle\!\rangle \\ & + \langle\!\langle \hat{\rho}, J \rangle\!\rangle + \langle\!\langle \hat{\sigma}, I \rangle\!\rangle\} \\ & \times \exp\{S_0(\phi, v) + S_0(\phi, \hat{\rho}) + S_0(\hat{\sigma}, \hat{\rho}) + S_0(\hat{\sigma}, v)\} \\ & \times \exp\{S_I(\phi, v, v) + S_I(\hat{\sigma}, \hat{\rho}, v)\} \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} \int & [\mathscr{D}\hat{\rho}][\mathscr{D}\hat{\sigma}][\mathscr{D}v_\alpha][\mathscr{D}\phi_\alpha] \\ & \equiv \int [\mathscr{D}\hat{\rho}(\mathbf{k}, w)][\mathscr{D}\hat{\sigma}(\mathbf{k}, w)] \prod_{\alpha=1}^{d} [\mathscr{D}v_\alpha(\mathbf{k}, w)][\mathscr{D}\phi_\alpha(\mathbf{k}, w)] \end{aligned} \tag{3.15a}$$

$$\langle\!\langle v_\alpha, l_\alpha \rangle\!\rangle \equiv \int^1 \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{dw}{2\pi} [v_\alpha(\mathbf{k}, w)\, l_\alpha(-\mathbf{k}, -w)] \tag{3.15b}$$

$$\begin{aligned} S_0(\phi, v) \equiv -i \int^1 & \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{dw}{2\pi} k^{2n} \left[\left(k^2 + \frac{iw}{v_0}\right)\right. \\ & \left.\times \delta_{\alpha\beta} + b_0 k_\alpha k_\beta\right] \phi_\alpha(\mathbf{k}, w)\, v_\beta(-\mathbf{k}, -w) \end{aligned} \tag{3.15c}$$

$$S_0(\phi, \hat{\rho}) \equiv -\frac{c_1}{v_0}\int^1 \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{dw}{2\pi} k^{2n}k_\alpha \phi_\alpha(\mathbf{k}, w)\,\hat{\rho}(-\mathbf{k}, -w) \tag{3.15d}$$

$$S_0(\hat{\sigma}, \hat{\rho}) \equiv -i\int^1 \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{dw}{2\pi} (iw)\,\hat{\sigma}(\mathbf{k}, w)\,\hat{\rho}(-\mathbf{k}, -w) \tag{3.15e}$$

$$S_0(\hat{\sigma}, v) \equiv -i\int^1 \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{dw}{2\pi} (-ik_\alpha)\,\hat{\sigma}(\mathbf{k}, w)\,v_\alpha(-\mathbf{k}, -w) \tag{3.15f}$$

$$S_I(\phi, v, v) \equiv \frac{(2\pi)^{d+1}}{v_0}\tilde{\int}^1 [q^{2n}s_\beta \delta_{\alpha\gamma}]\,\phi_\alpha(\mathbf{q}, B)\,v_\beta(\mathbf{r}, C)\,v_\gamma(\mathbf{s}, D) \tag{3.15g}$$

$$S_I(\hat{\sigma}, \hat{\rho}, v) \equiv -(2\pi)^{d+1}\tilde{\int}^1 [q_\gamma]\,\hat{\sigma}(\mathbf{q}, B)\,\hat{\rho}(\mathbf{q}, B)\,\hat{\rho}(\mathbf{r}, C)\,v_\gamma(\mathbf{s}, D) \tag{3.15h}$$

$$\tilde{\int}^1 \equiv \int^1 \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{dB}{2\pi}\frac{d^d\mathbf{r}}{(2\pi)^d}\frac{dC}{2\pi}\frac{d^d\mathbf{s}}{(2\pi)^d}\frac{dD}{2\pi}\delta^d(\mathbf{q}+\mathbf{r}+\mathbf{s}) \times \delta(B+C+D) \tag{3.15i}$$

In the above formulas, $\int^1$ denotes a cutoff of order 1 in momentum space generated by the "lattice structure."

In writing Eq. (3.15), I have neglected some nonlinear terms in Eq. (2.9), namely $\hat{\rho}/v_0(\partial v_\alpha/\partial t + v_\beta \partial_\beta v_\alpha)$. However, as I shall demonstrate shortly, these terms will turn out to be IR irrelevant in the RG sense. In next few paragraphs, I shall also explain how the requirement of locality, analyticity, and renormalizability places a constraint on the choice of n.

In writing down the equations of motions, namely Eqs. (2.9) and (2.10), we have been guided by the requirements of locality, analyticity, and Galilean invariance. Besides these requirements, however, the requirement of renormalizability places a strong constraint on the perturbative construction of the theory. It is observed that, regardless of the microscopic structure, the large-scale (in contrast to the molecular-scale) flow pattern of all fluids can be described by the same Navier–Stokes equation in terms of a few effective parameters relevant to the scale of observation. It is easy to see that the IR physics can be described by renormalizable theories, because the effects of nonrenormalizable terms are suppressed by the powers of the lattice cutoff. Renormalizable theories are short-distance-insensitive in the sense that they can be described in terms of a finite number of effective parameters relevant to the scale of observation without a detailed knowledge of the microscopic structure. Thus, a consistent approach to the scaling behavior of turbulence should be based on a renormalizable local field theory.

At this stage, the operator dimensionalities of the $\mathbf{v}$ field and the $\boldsymbol{\phi}$ field are indeterminate. However, it is easy to see that the linear part of the equations of motion of the $\boldsymbol{\phi}$ field is identical to that of the $\mathbf{v}$ field. [The equations of motion of the $\boldsymbol{\phi}$ field are obtained by differentiating the action in Eq. (3.15) with respect to the $\mathbf{v}$ field.] Hence, without loss of generality, we can assume that they are the same. It is, then, easy to see that $[v(\mathbf{x}, t)] \equiv [\phi(x, t)] = \Lambda^{d/2-n}$. From the discussion in Section 2, we know that Eqs. (2.9) and (2.10) are only approximate equations. The viscous force contains terms besides $\partial\partial v$. By power-counting (and because of the requirement of Galilean invariance), one finds that the most potentially IR important terms are of the following form: $\nabla \prod_{i=1}^{l} (\nabla_{\alpha_i} v_{\beta_i})$ with $l \geqslant 2$. The coefficients of these terms, in the continuum limit, will scale as $\{\Lambda^{n-1-d/2}\}^{l-1}$. Hence, if we require that these higher-order viscous terms be IR irrelevant both in 2 and 3 dimensions, we must restrict the choice of n to $n = \{0, 1, 2\}$. Different choices of n correspond to different expansion points of the perturbation theory. In Section 4, we shall see that, at the one-loop level, *the presence of a stable infrared physics uniquely determines n to be* 2.

Next, we have to determine the operator dimensionality of $\hat{\rho}$, and hence that of $\hat{\sigma}$. We choose $[\hat{\rho}(\mathbf{x}, t)] = \Lambda^{d/2-n+1}$ and $[\hat{\sigma}(\mathbf{x}, t)] = \Lambda^{d/2+n-1}$. In Appendix A, we demonstrate why this is the only choice. With all field operator dimensionalities determined, it is easy to see that the coefficients of $(\hat{\rho}/v_0)\, \partial v_\alpha/\partial t$ and $(\hat{\rho}/v_0)\, v_\beta \partial_\beta v_\alpha$ will scale in the continuum limit as $\Lambda^{n-1-d/2}$ and Λ^{2n-d}, respectively. Since we have $n \leqslant 2$ and $d \geqslant 2$, these two nonlinear terms are indeed IR irrelevant. The basic nonlinear term in the problem is $v_\beta \partial_\beta v_\alpha$, whose coefficient in the continuum limit scales as $\Lambda^{n+1-d/2}$.

We are now ready to map the theory with a curoff into a continuum field theory.[14] We are interested in the hydrodynamic, i.e., IR (with respect to the underlying lattice cutoff) behavior of the fluid. Following Brezin, Le Guillou, and Zinn-Justin (see ref. 14), we approach the hydrodynamic limit by keeping a_1, $\mathbf{k}'$ finite while letting $\Lambda \to \infty$ in the following rescaling:

$$\mathbf{k} \equiv \frac{a_1 \mathbf{k}'}{\Lambda}, \qquad a_1 > 0 \tag{3.16}$$

We also rescale the time and all fields accordingly in order to obtain a nontrivial field theory:

$$\mathbf{v} \equiv \zeta \mathbf{v}'; \qquad \boldsymbol{\phi} \equiv \eta \boldsymbol{\phi}'; \qquad \hat{\rho} \equiv a_3 \hat{\rho}'; \qquad \hat{\sigma} \equiv a_4 \hat{\sigma}' \tag{3.17}$$

$$w \equiv \frac{a_2 w'}{\Lambda^2}, \qquad a_2 > 0 \tag{3.18}$$

Then, in the primed coordinate system, with proper choices of ζ and η, the coupling constant of the nonlinear term will diverge as $\Lambda \to \infty$. However, this divergence of the coupling constant will cancel the divergence from the Feynman integrals in the perturbation series. The end result is a finite, renormalized theory.[14]

The choices of ζ and η are, to a degree, arbitrary. Since we have assumed that the operator dimensionalities of the $\mathbf{v}$ field and the $\boldsymbol{\phi}$ field are the same, we can set $\zeta = \eta$. In order to fix the form of the free propagator, we fix the coefficients of the following terms to be $-i$, -1, and $a_2/(a_1^2 v_0)$, respectively:

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{dw}{2\pi} k^{2n+2} \phi_\alpha(\mathbf{k}, w)\, v_\alpha(-\mathbf{k}, -w) \tag{3.19}$$

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{dw}{2\pi} k^{2n} k_\alpha \phi_\alpha(\mathbf{k}, w)\, \hat{\rho}(-\mathbf{k}, -w) \tag{3.20}$$

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{dw}{2\pi} w \hat{\sigma}(\mathbf{k}, w)\, \hat{\rho}(-\mathbf{k}, -w) \tag{3.21}$$

Then, the continuum bare theory of Eq. (3.15) is

$$\begin{aligned} Z \equiv \int & [\mathscr{D}\hat{\rho}_0][\mathscr{D}\hat{\sigma}_0][\mathscr{D}v_{0\alpha}][\mathscr{D}\phi_{0\alpha}] \\ & \times \exp\{\langle\!\langle v_{0\alpha}, l_{0\alpha} \rangle\!\rangle + \langle\!\langle \phi_{0\alpha}, h_{0\alpha} \rangle\!\rangle + \langle\!\langle \hat{\rho}_0, J_0 \rangle\!\rangle + \langle\!\langle \hat{\sigma}_0, I_0 \rangle\!\rangle\} \\ & \times \exp\{S_0(\phi_0, v_0) + S_0(\phi_0, \hat{\rho}_0) + S_0(\hat{\sigma}_0, \hat{\rho}_0) + S_0(\hat{\sigma}_0, v_0)\} \\ & \times \exp\{S_I(\phi_0, v_0, v_0) + S_I(\hat{\sigma}_0, \hat{\rho}_0, v_0)\} \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} \int & [\mathscr{D}\hat{\rho}_0][\mathscr{D}\hat{\sigma}_0][\mathscr{D}v_{0\alpha}][\mathscr{D}\phi_{0\alpha}] \\ & \equiv \int [\mathscr{D}\hat{\rho}_0(\mathbf{k}, w)][\mathscr{D}\hat{\sigma}_0(\mathbf{k}, w)] \prod_{\alpha=1}^{d} [\mathscr{D}v_{0\alpha}(\mathbf{k}, w)][\mathscr{D}\phi_{0\alpha}(\mathbf{k}, w)] \end{aligned} \tag{3.22a}$$

$$\langle\!\langle v_{0\alpha}, l_{0\alpha} \rangle\!\rangle \equiv \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{dw}{2\pi} [v_{0\alpha}(\mathbf{k}, w)\, l_{0\alpha}(-\mathbf{k}, -w)] \tag{3.22b}$$

$$\begin{aligned} S_0(\phi_0, v_0) \equiv -i \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{dw}{2\pi} k^{2n} \left[\left(k^2 + M \frac{iw}{v_0}\right) \delta_{\alpha\beta} + b_0 k_\alpha k_\beta\right] \\ \times \phi_{0\alpha}(\mathbf{k}, w)\, v_{0\beta}(-\mathbf{k}, -w) \end{aligned} \tag{3.22c}$$

$$S_0(\phi_0, \hat{\rho}_0) \equiv -\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{dw}{2\pi} k^{2n} k_\alpha \phi_{0\alpha}(\mathbf{k}, w)\, \hat{\rho}_0(-\mathbf{k}, -w) \tag{3.22d}$$

$$S_0(\hat{\sigma}_0, \hat{\rho}_0) \equiv -i \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{dw}{2\pi} \left[M \frac{iw}{v_0} \right] \hat{\sigma}_0(\mathbf{k}, w)\, \hat{\rho}_0(-\mathbf{k}, -w) \tag{3.22e}$$

$$S_0(\hat{\sigma}_0, v_0) \equiv -K\Lambda^2 \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{dw}{2\pi} k_\alpha \hat{\sigma}_0(\mathbf{k}, w)\, v_{0\alpha}(-\mathbf{k}, -w) \tag{3.22f}$$

$$S_I(\phi_0, v_0, v_0) \equiv \frac{(2\pi)^{d+1}}{v_0} N\Lambda^{n+1-d/2} \tilde{\int} [q^{2n} s_\beta \delta_{\alpha\gamma}]\, \phi_{0\alpha}(\mathbf{q}, B) \times v_{0\beta}(\mathbf{r}, C)\, v_{0\gamma}(\mathbf{s}, D) \tag{3.22g}$$

$$S_I(\hat{\sigma}_0, \hat{\rho}_0, v_0) \equiv -\frac{(2\pi)^{d+1}}{v_0} N\Lambda^{n+1-d/2} \tilde{\int} [q_\gamma]\, \hat{\sigma}_0(\mathbf{q}, B) \times \hat{\rho}_0(\mathbf{r}, C)\, v_{0\gamma}(\mathbf{s}, D) \tag{3.22h}$$

In the above equations, $M \equiv a_2/a_1^2$, $N \equiv a_1^{d/2-n-2} a_2^{1/2}$, and $K \equiv v_0^{-2} c_1 a_1^{-2}$. The subscript 0 denotes bare quantities. We have dropped the primed notation, even though we are now in the primed coordinate system. $\tilde{\int}$ is the same as in Eq. (3.15i), but with the upper momentum cutoff of order Λ. We obtain the continuum field theory by letting $\Lambda \to \infty$. The $\Lambda \to \infty$ limit will always be understood, but we shall not indicate this explicitly on every occasion.

Before we proceed to write down the perturbation theory for Eq. (3.22), we would like to point out some *generic* technical difficulty of the current formulation of the theory. This is easily illustrated in the following example. Let us consider some arbitrary classical equation of motion for the v field:

$$Kv + V(v) = 0 \tag{3.23}$$

where K is some invertible linear operator and $V(v)$ denotes all nonlinear terms of the v field. If we average over all solutions of Eq. (3.23) with equal weight, then the generating functional of all correlation functions is:

$$Z = \int [\mathscr{D}\phi][\mathscr{D}v] \exp\left\{ \int lv + \int h\phi \right\} \exp\left\{ -\int i\phi[Kv + V(v)] \right\}$$
$$= \exp\left\{ -i \int \frac{\delta}{\delta h} V\left[\frac{\delta}{\delta l} \right] \right\} \exp\left\{ -i \int lK^{-1}h \right\} \tag{3.24}$$

It is easy to see that $\langle vv \rangle \equiv (\delta^2/\delta l^2)\, Z|_{l=h=0} = 0$, because each vertex in $V[\delta/\delta l]$ brings down too many powers of h, which cannot be canceled by a single differentiation of $\delta/\delta h$. The same conclusion holds for all other physical correlation functions (or moments) of the v field. On the other hand, physical intuition tells us that, for complex nonlinear equations, there exist nontrivial results. Obviously, we cannot proceed with the perturbation theory naively as written in Eq. (3.22). We have to shift the field to locate the proper expansion point.

In order to find a nontrivial expansion point consistently, we find it convenient to use $\tilde{Z}$, as defined in the following. The reasons for this will become clear momentarily:

$$\begin{aligned}\tilde{Z} \equiv \int & [\mathscr{D}\hat{\rho}_0][\mathscr{D}\hat{\sigma}_0][\mathscr{D}v_{0\alpha}][\mathscr{D}\phi_{0\alpha}] \\ & \times \exp\left\{\langle\!\langle v_{0\alpha} + \frac{i}{2}\phi_{0\alpha}, l_{0\alpha}\rangle\!\rangle + \langle\!\langle \phi_{0\alpha}, h_{0\alpha}\rangle\!\rangle + \langle\!\langle \hat{\rho}_0, J_0\rangle\!\rangle + \langle\!\langle \hat{\sigma}_0, I_0\rangle\!\rangle\right\} \\ & \times \exp\{S_0(\phi_0, v_0) + S_0(\phi_0, \hat{\rho}_0) + S_0(\hat{\sigma}_0, \hat{\rho}_0) + S_0(\hat{\sigma}_0, v_0)\} \\ & \times \exp\{S_I(\phi_0, v_0, v_0) + S_I(\hat{\sigma}_0, \hat{\rho}_0, v_0)\} \end{aligned} \tag{3.25}$$

The only difference between Eq. (3.25) and Eq. (3.22) is the source term. One can regard Eq. (3.25) as an Ansatz, which, in contrast to Eq. (3.24), will generate a nontrivial perturbation theory. The purpose of considering this "shifted" new theory will become clear shortly—it is possible to shift in this manner because the operator dimensionalities of $\mathbf{v}$ and $\boldsymbol{\phi}$ fields are the same.

Using the results in Appendix B and C, and after some tedious algebra, it is possible to rewrite $\tilde{Z}$ in the following way:

$$\begin{aligned}\tilde{Z} \equiv \int & [\mathscr{D}\hat{\rho}_0][\mathscr{D}\hat{\sigma}_0][\mathscr{D}u_{0\alpha}][\mathscr{D}\phi_{0\alpha}] \\ & \times \exp\{\langle\!\langle u_{0\alpha}, l_{0\alpha}\rangle\!\rangle + \langle\!\langle \phi_{0\alpha}, h_{0\alpha}\rangle\!\rangle + \langle\!\langle \hat{\rho}_0, J_0\rangle\!\rangle + \langle\!\langle \hat{\sigma}_0, I_0\rangle\!\rangle\} \\ & \times \exp\left\{S_0(\phi_0, \phi_0) + S_0(\phi_0, u_0) + S_0(\phi_0, \hat{\rho}_0)\right. \\ & \left. + S_0(\hat{\sigma}_0, \hat{\rho}_0) + S_0\left(\hat{\sigma}_0, u_0 - \frac{i}{2}\phi_0\right)\right\} \\ & \times \exp\left\{S_I(\phi_0, u_0, u_0) + S_I(\phi_0, \phi_0, u_0) + S_I(\phi_0, \phi_0, \phi_0)\right. \\ & \left. + S_I\left(\hat{\sigma}_0, \hat{\rho}_0, u_0 - \frac{i}{2}\phi_0\right)\right\} \end{aligned} \tag{3.26}$$

One can show that there is one-to-one correspondence between Feynman graphs of Eqs. (3.25) and (3.26). One also notices that $u_{0\alpha}(\mathbf{x}, t)$ is a *real* field, not equal to $v_{0\alpha}(\mathbf{x}, t) + (i/2)\,\phi_{0\alpha}(\mathbf{x}, t)$, even though the relation is highly suggestive. In Eq. (3.26), we have used the following symbols:

$$S_0(\phi_0, \phi_0) \equiv -\frac{1}{2}\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{dw}{2\pi}\, k^{2n}\left[\left(k^2 + Z_\nu \frac{iw}{\nu}\right)\delta_{\alpha\beta} + b_0 k_\alpha k_\beta\right] \times \phi_{0\alpha}(\mathbf{k}, w)\, \phi_{0\beta}(-\mathbf{k}, -w) \quad (3.26a)$$

$$S_0(\phi_0, u_0) \equiv -i\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{dw}{2\pi}\, k^{2n}\left[\left(k^2 + Z_\nu \frac{iw}{\nu}\right)\delta_{\alpha\beta} + b_0 k_\alpha k_\beta\right] \times \phi_{0\alpha}(\mathbf{k}, w)\, u_{0\beta}(-\mathbf{k}, -w) \quad (3.26b)$$

$$S_0(\phi_0, \hat{\rho}_0) \equiv -\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{dw}{2\pi}\, k^{2n} k_\alpha \phi_{0\alpha}(\mathbf{k}, w)\, \hat{\rho}_0(-\mathbf{k}, -w) \quad (3.26c)$$

$$S_0(\hat{\sigma}_0, \hat{\rho}_0) \equiv -i\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{dw}{2\pi}\left[Z_\nu \frac{iw}{\nu}\right]\hat{\sigma}_0(\mathbf{k}, w) \times \hat{\rho}_0(-\mathbf{k}, -w) \quad (3.26d)$$

$$S_0\left(\hat{\sigma}_0, u_0 - \frac{i}{2}\phi_0\right) \equiv -m_0\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{dw}{2\pi}\, k_\alpha \hat{\sigma}_0(\mathbf{k}, w) \times \left[u_{0\alpha}(-\mathbf{k}, -w) - \frac{i}{2}\phi_{0\alpha}(-\mathbf{k}, -w)\right] \quad (3.26e)$$

$$S_I(\phi_0, u_0, u_0) \equiv -\frac{g_0}{2!}\tilde{\int} K_{\alpha\beta\gamma}\phi_{0\alpha}(\mathbf{q}, B)\, u_{0\beta}(\mathbf{r}, C)\, u_{0\gamma}(\mathbf{s}, D) \quad (3.26f)$$

$$S_I(\phi_0, \phi_0, u_0) \equiv -\frac{g_0}{2(2!)}\tilde{\int} L_{\alpha\beta\gamma}\phi_{0\alpha}(\mathbf{q}, B) \times \phi_{0\beta}(\mathbf{r}, C)\, u_{0\gamma}(\mathbf{s}, D) \quad (3.26g)$$

$$S_I(\phi_0, \phi_0, \phi_0) \equiv -\frac{g_0}{4(3!)}\tilde{\int} M_{\alpha\beta\gamma}\phi_{0\alpha}(\mathbf{q}, B) \times \phi_{0\beta}(\mathbf{r}, C)\, \phi_{0\gamma}(\mathbf{s}, D) \quad (3.26h)$$

$$S_I\left(\hat{\sigma}_0, \hat{\rho}_0, u_0 - \frac{i}{2}\phi_0\right) \equiv -g_0\tilde{\int}[q_\gamma]\,\hat{\sigma}_0(\mathbf{q}, B)\,\hat{\rho}_0(\mathbf{r}, C) \times\left[u_{0\gamma}(\mathbf{s}, D) - \frac{i}{2}\phi_{0\gamma}(\mathbf{s}, D)\right] \quad (3.26i)$$

In the above equations, $M/v_0 \equiv Z_v/v$, $[(2\pi)^{d+1}/v_0]\, N\Lambda^{n+1-d/2} \equiv g_0$, and $K\Lambda^2 \equiv m_0$.

In order to write down the Feynman rules in the momentum space, we have symmetrized the interaction vertices:

$$K_{\alpha\beta\gamma} \equiv -q^{2n} s_\beta \delta_{\alpha\gamma} - q^{2n} r_\gamma \delta_{\alpha\beta} \tag{3.27}$$

$$L_{\alpha\beta\gamma} \equiv iq^{2n} s_\beta \delta_{\alpha\gamma} + iq^{2n} r_\gamma \delta_{\alpha\beta} + ir^{2n} s_\alpha \delta_{\beta\gamma} + ir^{2n} q_\gamma \delta_{\alpha\beta} \tag{3.28}$$

$$M_{\alpha\beta\gamma} \equiv (q^{2n} s_\beta + s^{2n} q_\beta)\, \delta_{\alpha\gamma} + (r^{2n} s_\alpha + s^{2n} r_\alpha)\, \delta_{\beta\gamma} + (q^{2n} r_\gamma + r^{2n} q_\gamma)\, \delta_{\alpha\beta} \tag{3.29}$$

The main reason for considering the new theory $\tilde{Z}$ is as follows. Let us denote the original theory Z, i.e., Eq. (3.22), by subscript 1 and the new theory $\tilde{Z}$, i.e., Eq. (3.25) or Eq. (3.26), by subscript 2. It is obvious that

$$\langle v_0 v_0 \rangle_1 \equiv \langle u_0 u_0 - iu_0 \phi_0 - \tfrac{1}{4} \phi_0 \phi_0 \rangle_2 \tag{3.30}$$

$$\langle \phi_0 \phi_0 \rangle_1 \equiv \langle \phi_0 \phi_0 \rangle_2 \tag{3.31}$$

Using these two relations, one can unambiguously identify the leading IR scaling behavior of the velocity–velocity correlation function once we locate the IR stable fixed point in the theory $\tilde{Z}$. It is possible to verify that the right-hand side of Eq. (3.30) is nonzero, order by order in the coupling constants.

The inductive proof of the multiplicative renormalizability of Eq. (3.26) is quite involved and will not be elaborated here. Suffice it to say that one can follow the usual Zimmermann's forest-formula[15] for this theory as well. Let us introduce the following set of wave-function renormalization constants:

$$u_{0\alpha}(\mathbf{k}, w) \equiv Z^{1/2} u_\alpha(\mathbf{k}, w); \qquad l_{0\alpha}(\mathbf{k}, w) \equiv Z^{-1/2} l_\alpha(\mathbf{k}, w) \tag{3.32}$$

$$\phi_{0\alpha}(\mathbf{k}, w) \equiv Z_\phi^{1/2} \phi_\alpha(\mathbf{k}, w); \qquad h_{0\alpha}(\mathbf{k}, w) \equiv Z_\phi^{-1/2} h_\alpha(\mathbf{k}, w) \tag{3.33}$$

$$\hat{\rho}_0(\mathbf{k}, w) \equiv Z_\rho^{1/2} \hat{\rho}(\mathbf{k}, w); \qquad J_0(\mathbf{k}, w) \equiv Z_\rho^{-1/2} J(\mathbf{k}, w) \tag{3.34}$$

$$\hat{\sigma}_0(\mathbf{k}, w) \equiv Z_\sigma^{1/2} \hat{\sigma}(\mathbf{k}, w); \qquad I_0(\mathbf{k}, w) \equiv Z_\sigma^{-1/2} I(\mathbf{k}, w) \tag{3.35}$$

From the general structure of the free propagators in Appendix C, one can see that calculation in this field theory is very complicated. In order to simplify this presentation, we concentrate on the "critical" theory. The renormalized coupling constants m and f (in Appendix C) have a dimension of Λ^2. They will set a length scale in the problem. Since we are looking for the critical theory, we set $m = f = 0$. The renormalized b and e (in Appendix C) are dimensionless. However, they are related to the ratio of the second viscosity coefficient to the first viscosity coefficient. Since we are interested in fully-developed turbulence, where the nonlinear term dominates the viscous-dissipative term, it is physically more satisfying to set $b=e=0$. It is not clear whether there are other fixed points with non-zero values of b and e, nor is it clear what their physical significance would be.

Based on the above considerations, we have the following four free propagators in the "critical" theory:

$$\frac{(-i)\,k_\alpha l_\alpha(-\mathbf{k},\,-w)\,I(\mathbf{k},\,w)}{(k^2-iw/v)(w/v+i\eta^+)};\qquad \frac{I(\mathbf{k},\,w)\,J(-\mathbf{k},\,-w)}{w/v+i\eta^+} \tag{3.36}$$

$$\frac{(-i)\,h_\alpha(\mathbf{k},\,w)\,l_\alpha(-\mathbf{k},\,-w)}{k^{2n}(k^2-iw/v)};\qquad \frac{1}{2}\frac{l_\alpha(\mathbf{k},\,w)\,l_\alpha(-\mathbf{k},\,-w)}{k^{2n-2}(k^4+w^2/v^2)} \tag{3.37}$$

The η^+ prescription for the w-integration is explained in Appendix C.

From the structure of the above free propagators in the "critical" theory, we conclude that the vertices in the last term of Eq. (3.26), namely $S_I(\hat{\sigma}_0, \hat{\rho}_0, u_0-\frac{1}{2}i\phi_0)$, do not play a role in determining the wave-function renormalization of the $\mathbf{u}$ and $\boldsymbol{\phi}$ fields. It is also clear that each loop integration will contribute a v factor from the frequency integration. Let L denote the number of loops; I denotes the number of internal lines; E denotes the number of external points; and n_3 denotes the number of vertices. From $L=I-(n_3-1)$ and $I=\frac{1}{2}(3n_3-E)$, we obtain $L=n_3/2+(1-E/2)$. Hence, if we denote the renormalized coefficients of the nonlinear terms as $g_i/\sqrt{v}$, the dependence of n-point Green's functions, G_N, on v will be through the combination w/v and an overall multiplicative factor $v^{1-N/2}$. Here, we denote g_1, g_2, and g_3 as the renormalized coupling constants for the vertices denoted by $K_{\alpha\beta\gamma}$, $L_{\alpha\beta\gamma}$, and $M_{\alpha\beta\gamma}$, respectively, in Eq. (3.26).

Let us concentrate on the 2-point Green's function. The renormalization group equation for G_2 is

$$\left[\mu\frac{\partial}{\partial\mu}+\beta_j\frac{\partial}{\partial g_j}+(-\gamma_v)\,v\frac{\partial}{\partial v}+\gamma\right]G_2\left(k;\frac{w}{v},\,g_i,\,\mu\right)=0 \tag{3.38}$$

where G_2 is the 2-point Green's function of the $\mathbf{u}$ field. If we were to consider the 2-point Green's function of the $\boldsymbol{\phi}$ field, we would just replace γ by γ_ϕ in the above equation. Here, μ is the usual "unit of mass," $\beta_j\equiv\mu\,dg_j/d\mu$, $\gamma_v\equiv-\mu\,d\ln Z_v/d\mu$, $\gamma\equiv\mu\,d\ln Z/d\mu$, and $\gamma_\phi\equiv\mu\,d\ln Z_\phi/d\mu$. Since there is no multiplicative prefactor of v for G_2, we can rewrite Eq. (3.38) as

$$\left[\mu\frac{\partial}{\partial\mu}+\beta_j\frac{\partial}{\partial g_j}+(\gamma_v)\,w\frac{\partial}{\partial w}+\gamma\right]G_2\left(k;\frac{w}{v},\,g_i,\,\mu\right)=0 \tag{3.39}$$

At the fixed points $\{g_i^*\}$ where $\{\beta_i(g_j^*)=0\}$, the most general solution of Eq. (3.39) is

$$G_2\left(k; \frac{w}{v}, g_i^*, \mu\right) = \mu^{-\gamma} F_2\left(k; \frac{w}{\mu^{\gamma_v} v}, g_i^*\right) \tag{3.40}$$

with F_2 some unknown function of three variables. It is, easy to show that

$$\begin{aligned} E(k) &\equiv k^{d-1} \int_{-\infty}^{\infty} dw\, G_2\left(k; \frac{w}{v}, g_i^*, \mu\right) \\ &\approx k^{d-5+\gamma_a-\gamma_v} \mu^{-\gamma_a+\gamma_v} v \int_{-\infty}^{\infty} dx\, F_2(1; x, g_i^*) \end{aligned} \tag{3.41}$$

where $\gamma_a = \gamma$, or γ_ϕ, or $1/2(\gamma + \gamma_\phi)$, depending on the degree of singularity of the leading scaling behavior of $\langle uu \rangle$, or $\langle \phi\phi \rangle$, or $\langle u\phi \rangle$ at the fixed points. This is easily seen from Eq. (3.30).

4. ONE-LOOP RESULTS

From Appendix C, it is easily seen that some free propagators are "directional" in the w-space. This is true even if we are not considering the "critical" theory. As a consequence, some loops vanish due to the w-integration, even though the power-counting indicates that they are divergent. Detailed analysis also shows that the 2-point function $\langle \phi\phi \rangle$ vanishes *to all orders* in the coupling constants.

We have already learned from Section 3 that the requirement of renormalizability constrains n to be 0, 1, or 2. If $n=0$, the upper critical dimension of the system will be 2. However, this would mean that at 3 dimensions, the nonlinear term is unimportant and irrelevant. This contradicts the experimental evidence. Obviously, we are expanding around the wrong expansion point in this case. Hence, the possibility that $n=0$ is ruled out.

Next, let us consider the case with $n=1$. Using *dimensional regularization* and *minimal subtraction*, I find, at the one-loop level, the following dimensional poles of the self-energy (for the "critical" theory):

$$\Sigma(\phi-\phi) \approx \frac{1}{(4\pi)^2} \cdot \frac{1}{\varepsilon} \cdot \delta_{\alpha\beta} q^4 \left[-\frac{3}{8} g_1^2 + \frac{3}{2} g_1 g_2 \right] \tag{4.1}$$

$$\Sigma(\phi-u) \approx \frac{i}{(4\pi)^2} \cdot \frac{1}{\varepsilon} \cdot \delta_{\alpha\beta} q^4 \left[-\frac{3}{8} g_1^2 + \frac{3}{4} g_1 g_2 \right] \tag{4.2}$$

In Appendix D, I list all the integrals appearing in the one-loop calculation. It is, then, easy to see that the wave-function renormalization constants, at the one-loop level, are

$$Z_\phi \approx 1 + \frac{1}{(4\pi)^2} \cdot \frac{1}{\varepsilon} \cdot \left[-\frac{3}{8} g_1^2 + \frac{3}{2} g_1 g_2 \right] \tag{4.3}$$

$$Z \approx 1 + \frac{1}{(4\pi)^2} \cdot \frac{1}{\varepsilon} \cdot \left[-\frac{3}{8} g_1^2 \right] \tag{4.4}$$

$$Z_v \approx 1 + \frac{1}{(4\pi)^2} \cdot \frac{1}{\varepsilon} \cdot \left[\frac{3}{8} g_1^2 - \frac{3}{4} g_1 g_2 \right] \tag{4.5}$$

At the one-loop level, the dimensional poles of the vertex correction magically cancel out. After some simple algebra, we obtain the following β-functions:

$$\beta_1 \approx -\frac{\varepsilon}{2} g_1 + \frac{1}{(4\pi)^2} \left[\frac{3}{8} g_1^3 - \frac{3}{8} g_1^2 g_2 \right] \tag{4.6}$$

$$\beta_2 \approx -\frac{\varepsilon}{2} g_2 + \frac{1}{(4\pi)^2} \left[\frac{3}{8} g_1^2 g_2 - \frac{9}{8} g_1 g_2^2 \right] \tag{4.7}$$

$$\beta_3 \approx -\frac{\varepsilon}{2} g_3 + \frac{1}{(4\pi)^2} \left[\frac{3}{8} g_1^2 g_3 - \frac{15}{8} g_1 g_2 g_3 \right] \tag{4.8}$$

The fixed points of Eqs. (4.6)–(4.8) are

$$g_1 = g_2 = g_3 = 0 \tag{4.9}$$

$$g_1 = \pm 4\pi \left(\frac{4\varepsilon}{3} \right)^{1/2}, \qquad g_2 = 0, \qquad g_3 \text{ arbitrary} \tag{4.10}$$

Unfortunately, none of these fixed points is IR stable. From Eqs. (3.26f)–(3.26h) we have $g_{01} = 2g_{02} = 4g_{03} > 0$. Then g_1, g_2, and g_3 will be in the same region of the (g_1, g_2, g_3) space. Numerical integration of the β-functions indicates that with $g_1(\mu_0) \approx 2g_2(\mu_0) \approx 4g_3(\mu_0) > 0$, g_2 will always grow in an unbounded manner in the IR limit. The flow diagram near the g_1 axis in the (g_1, g_2) plane bears some resemblance to that of the scalar electrodynamics.[16] *Hence, there is no nontrivial IR stable fixed point.* At the one-loop level, therefore, we have to rule out the possible choice of $n = 1$. Figure 1 presents a typical β-function flow pattern in the (g_1, g_2) plane near the unstable fixed point in Eq. (4.10).

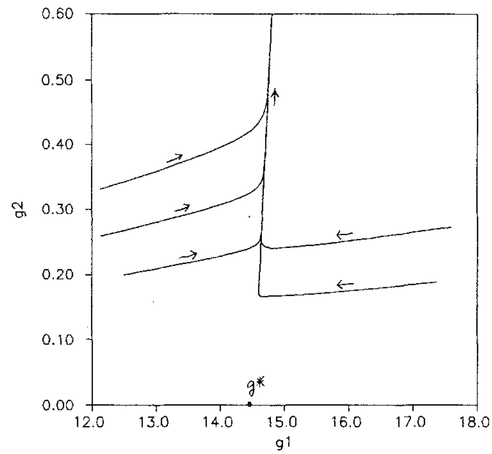

Fig. 1. β-function flow for the case with $n=1$ (and $\varepsilon=1$).

Finally, we consider the case of $n=2$. With $n=2$, the upper critical dimension of the system is $d_c=6$. Using *dimensional regularization* and *minimal subtraction*, I find, at the one-loop level, the following dimensional poles of the self-energy (for the "critical" theory):

$$\Sigma(\phi-\phi)\approx\frac{1}{(4\pi)^3}\cdot\frac{1}{\varepsilon}\cdot\delta_{\alpha\beta}q^6\left[-\frac{5}{6}g_1g_2+\frac{5}{6}g_1g_3\right] \tag{4.11}$$

$$\Sigma(\phi-u)\approx\frac{i}{(4\pi)^3}\cdot\frac{1}{\varepsilon}\cdot\delta_{\alpha\beta}q^6\left[-\frac{5}{24}g_1^2+\frac{5}{12}g_1g_2\right] \tag{4.12}$$

The corresponding wave-function renormalization constants are

$$Z_\phi\approx 1+\frac{1}{(4\pi)^3}\cdot\frac{1}{\varepsilon}\cdot\left[-\frac{5}{6}g_1g_2+\frac{5}{6}g_1g_3\right] \tag{4.13}$$

$$Z\approx 1+\frac{1}{(4\pi)^3}\cdot\frac{1}{\varepsilon}\cdot\left[-\frac{5}{12}g_1^2+\frac{5}{3}g_1g_2-\frac{5}{6}g_1g_3\right] \tag{4.14}$$

$$Z_v\approx 1+\frac{1}{(4\pi)^3}\cdot\frac{1}{\varepsilon}\cdot\left[\frac{5}{24}g_1^2-\frac{5}{12}g_1g_2\right] \tag{4.15}$$

Again, at the one-loop level, the dimensional poles of the vertex correction vanish. The β-functions are then given by

$$\beta_1 \approx -\frac{\varepsilon}{2} g_1 + \frac{1}{(4\pi)^3}\left[\frac{5}{16} g_1^3 - \frac{25}{24} g_1^2 g_2 + \frac{5}{12} g_1^2 g_3\right] \tag{4.16}$$

$$\beta_2 \approx -\frac{\varepsilon}{2} g_2 + \frac{1}{(4\pi)^3}\left[\frac{5}{48} g_1^2 g_2 + \frac{5}{24} g_1 g_2^2 - \frac{5}{12} g_1 g_2 g_3\right] \tag{4.17}$$

$$\beta_3 \approx -\frac{\varepsilon}{2} g_3 + \frac{1}{(4\pi)^3}\left[-\frac{5}{48} g_1^2 g_3 + \frac{35}{24} g_1 g_2 g_3 - \frac{5}{4} g_1 g_3^2\right] \tag{4.18}$$

The zeros of the above equations consist of two isolated fixed points and a fixed line:

$$g_1 = g_2 = g_3 = 0 \tag{4.19}$$

$$g_1 = \pm\left[\frac{8\varepsilon}{5}(4\pi)^3\right]^{1/2}, \qquad g_2 = g_3 = 0 \tag{4.20}$$

$$g_1 = a \pm \left[\frac{a^2 + 18\varepsilon(4\pi)^3}{5}\right]^{1/2}, \qquad g_2 = \frac{5a \pm [a^2 + 18\varepsilon(4\pi)^3/5]^{1/2}}{6},$$
$$g_3 = a, \qquad a \in \mathscr{R} \tag{4.21}$$

The fixed points in Eqs. (4.19) and (4.20) have at least two negative eigenvalues. They are, therefore, IR *unstable*. At the fixed points of Eq. (4.21), the derivative of the β-functions has two positive (for small a) and one flat direction. One has to numerically integrate the β-functions to determine whether the coupling constants corresponding to the physical theory will be attracted to this fixed line. We have tested that, for $g_1(\mu_0) \approx 2g_2(\mu_0) \approx 4g_3(\mu_0) > 0$, and $g_3(\mu_0)$ arbitrary, the flow of the coupling constants is always toward the fixed line in Eq. (4.21) in the IR limit. *Hence, the physical theory is indeed attracted toward the fixed line.* It is also easy to check that when $a > a_c \approx 29.88\sqrt{\varepsilon}$, one of the eigenvalues of the fixed points in Eq. (4.21) will become negative. Therefore, the physically accessible fixed points are restricted to $0 < a < a_c \approx 29.88\sqrt{\varepsilon}$. This is consistent within the framework of the ε expansion: at the one-loop level, the fixed-point value of the coupling constant is of the order of ε. (We are considering a theory similar to ϕ^3, and therefore the fixed-point value of the coupling constant is of the order of $\sqrt{\varepsilon}$.)

Along the fixed line of Eq. (4.21), we have

$$\gamma - \gamma_v \approx \frac{1}{(4\pi)^3}\left[\frac{5}{24} g_1^2 - \frac{5}{4} g_1 g_2 + \frac{5}{6} g_1 g_3\right] = 0 \tag{4.22}$$

$$\gamma_\phi - \gamma_v \approx \frac{1}{(4\pi)^3}\left[-\frac{5}{24} g_1^2 + \frac{5}{4} g_1 g_2 - \frac{5}{6} g_1 g_3\right] = 0 \tag{4.23}$$

From Eq. (3.41), we obtain

$$E(k) \approx k^{-2} \qquad \text{in 3 dimensions} \tag{4.24}$$

$$E(k) \approx k^{-3} \qquad \text{in 2 dimensions} \tag{4.25}$$

I would like to emphasize that the above scaling behavior is at the *nontrivial fixed line*, where the renormalized nonlinear terms dominate. The fact that there is no "anomalous" dimension at the first order in ε is a pure coincidence. We have a similar situation in ϕ^4 theory.

5. SUMMARY AND DISCUSSION

In summary, I have proposed a new approach for calculating the scaling exponents of the freely-decaying turbulence. At the one-loop level, this approach predicts two independent exponents, to within 10% of the best available experimental values. In this approach, one does not have to consider separately the energy cascade in 3 dimensions and enstrophy cascade in 2 dimensions in order to derive the correct energy spectrum, nor does one have to worry about the so-called "intermittency correction."[(2)] Assuming *only* that all relevant information is contained in the equations of motion, and treating these equations *consistently* from the perspective of quantum field theory, one is able to compute all exponents without any *ad hoc* assumptions.

Even though we have obtained very reasonable answers at the one-loop level, there remain some unresolved questions. Does the ε expansion make sense when the operator dimensionality of the field becomes negative (when $d < 4$ for $n = 2$)? One encounters the same problem in the Gaussian-random-force approach.[(5)] Will the fixed line in Eq. (4.21) be resolved into a fixed point by higher-loop calculations? What will happen to the marginal operator corresponding to the zero eigenvalue in Eq. (4.21)? A much harder problem is the Borel summability of the ε expansion. I hope this article will stimulate further interest in these problems.

APPENDIX A

Looking back at the quadratic part of Eq. (3.15), one realizes that there are two other possible choices of the operator dimensionalities for the $\hat{\rho}$ and $\hat{\sigma}$ fields. We discuss them now.

The first one is to set $[\hat{\sigma}(\mathbf{x}, t)] \equiv \Lambda^{d/2+n+1}$ and $[\hat{\rho}(\mathbf{x}, t)] \equiv \Lambda^{d/2-n-1}$. However, this will result in a nonrenormalizable theory. As discussed in Section 2, in the expansion of P in terms of local and analytical functions of $\hat{\rho}$ and $\mathbf{v}$, the following term will appear in the action:

$$\int d^d\mathbf{x}\, dt\, \phi_\alpha(\mathbf{x}, t)\, \nabla^{2n} \partial_\alpha (\hat{\rho})^m \qquad \text{with} \quad m \geqslant 2 \tag{A1}$$

Simple power-counting shows that the coefficient of this term will scale in the continuum limit as $\Lambda^{2+(m-1)(n+1-d/2)}$. This is more IR important than the basic nonlinear term kept in the equations of motion, namely

$$\int d^d\mathbf{x}\, dt\, \phi_\alpha(\mathbf{x}, t)\, \nabla^{2n} v_\beta \partial_\beta v_\alpha \tag{A2}$$

whose coefficient in the continuum limit only scales as $\Lambda^{n+1-d/2}$. This contradicts our notion that the turbulence of near-incompressible fluid results from the dominance of the $v\partial v$ nonlinear term. Furthermore, with m big enough, there are infinitely many relevant terms in the equations of motion. This results in a logically inconsistent theory. In fact, the expansion adopted in ref. 17 is precisely this one. Even though all these relevant operators do not contribute at the first order in ε (because they all carry to many "legs"), it does not mean that the answer can be trusted. As explained in the second paragraph after Eq. (3.15), a consistent approach to the scaling behavior of turbulence should be based on a renormalizable local field theory.

The only other possible choice of the operator dimensionalities of $\hat{\rho}$ and $\hat{\sigma}$ fields is to set $[\hat{\rho}(\mathbf{x}, t)] \equiv \Lambda^{d/2-n+1}$ and $[\hat{\sigma}(\mathbf{x}, t)] \equiv \Lambda^{d/2+n+1}$. Simple power-counting indicates that the resulting theory is identical to the incompressible turbulence. The effect of $\hat{\rho}$ and $\hat{\sigma}$ fields is to act as the auxiliary fields enforcing the incompressibility condition: $\partial_\alpha v_\alpha = 0$ and $\partial_\alpha \phi_\alpha = 0$.[8] However, the renormalization of the incompressible turbulence presents many peculiar features. At the first order in ε, there is no strong evidence in favor of the choice of $n = 2$ (except the fact that $n = 2$ fits the experimental results better). Furthermore, the fixed-point values of the coupling constants are not necessarily of the order of $\sqrt{\varepsilon}$. These peculiarities reflect the unphysical features of the incompressible turbulence—disturbances are propagated with infinite speed. Hence, we discard this choice. As we demonstrate in Section 4, all these "diseases" disappear within the framework of near-incompressible turbulence.

APPENDIX B

In this Appendix, we show

$$\begin{aligned}
\tilde{Z}_0 \equiv \int & [\mathscr{D}\hat{\rho}][\mathscr{D}\hat{\sigma}][\mathscr{D}v_\alpha][\mathscr{D}\phi_\alpha] \\
& \times \exp\{\langle\langle v_\alpha + ia\phi_\alpha, l_\alpha\rangle\rangle + \langle\langle \phi_\alpha, h_\alpha\rangle\rangle + \langle\langle \hat{\rho}, J\rangle\rangle + \langle\langle \hat{\sigma}, I\rangle\rangle\} \\
& \times \exp\left\{-i\int k^{2n}\left[\left(k^2 + \frac{iw}{v}\right)\delta_{\alpha\beta} + bk_\alpha k_\beta\right]\phi_\alpha(\mathbf{k}, w)\, v_\beta(-\mathbf{k}, -w)\right\} \\
& \times \exp\left\{-\int [k^{2n}k_\alpha\phi_\alpha(\mathbf{k}, w)\,\hat{\rho}(-\mathbf{k}, -w) + mk_\alpha\hat{\sigma}(\mathbf{k}, w)\, v_\alpha(-\mathbf{k}, -w)]\right\} \\
& \times \exp\left\{-i\int\left(\frac{iw}{v}\right)\hat{\sigma}(\mathbf{k}, w)\,\hat{\rho}(-\mathbf{k}, -w)\right\} \\
= \exp & \left\{\int(-i)\frac{[k^2 + iw/v + bk^2 - im(v/w)\,k^2]\,vk_\alpha}{[k^4 + w^2/v^2 + ck^2]\,w}\, l_\alpha(-\mathbf{k}, -w)\, I(\mathbf{k}, w)\right\} \\
& \times \exp\left\{\int\left[\frac{v}{w} + (-i)\frac{[k^2 + iw/v + bk^2 - im(v/w)\,k^2]\,mv^2k^2}{[k^4 + w^2/v^2 + ck^2]\,w^2}\right]\right. \\
& \left.\times I(\mathbf{k}, w)\, J(-\mathbf{k}, -w)\right\} \\
& \times \exp\left\{\int a\frac{[k^2 - iw/v + bk^2 + im(v/w)\,k^2]\,mvk_\alpha}{k^{2n}[k^4 + w^2/v^2 + ck^2]\,w}\, l_\alpha(-\mathbf{k}, -w)\, J(\mathbf{k}, w)\right\} \\
& \times \exp\left\{\int(-i)\frac{[k^2 + iw/v + bk^2 - im(v/w)k^2]\,mvk_\alpha}{k^{2n}[k^4 + w^2/v^2 + ck^2]\,w} h_\alpha(\mathbf{k}, w)\, J(-\mathbf{k}, -w)\right\} \\
& \times \exp\left\{\int i\frac{[c(k^2 + iw/v) + (k^4 + w^2/v^2)(-b + im(v/w))]\,k_\alpha k_\beta}{k^{2n}[k^4 + w^2/v^2][k^4 + w^2/v^2 + ck^2]}\right. \\
& \left.\times h_\alpha(\mathbf{k}, w)\, l_\beta(-\mathbf{k}, -w)\right\} \\
& \times \exp\left\{\int(-a)\frac{[ck^2 - b(k^4 + w^2/v^2)]\,k_\alpha k_\beta}{k^{2n}[k^4 + w^2/v^2][k^4 + w^2/v^2 + ck^2]}\, l_\alpha(\mathbf{k}, w)\, l_\beta(-\mathbf{k}, -w)\right\} \\
& \times \exp\left\{\int\left[\frac{(-i)\,h_\alpha(\mathbf{k}, w)\, l_\alpha(-\mathbf{k}, -w)}{k^{2n}(k^2 - iw/v)} + a\frac{l_\alpha(\mathbf{k}, w)\, l_\alpha(-\mathbf{k}, -w)}{k^{2n-2}(k^4 + w^2/v^2)}\right]\right\} \qquad \text{(B1)}
\end{aligned}$$

where

$$\langle\langle v_\alpha + ia\phi_\alpha, l_\alpha\rangle\rangle \equiv \int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{dw}{2\pi}\, l_\alpha(-\mathbf{k}, -w)[v_\alpha(\mathbf{k}, w) + ia\phi_\alpha(\mathbf{k}, w)] \qquad \text{(B2)}$$

$$\int \equiv \int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{dw}{2\pi} \qquad \text{and} \qquad c \equiv k^2\left(b^2 + 2b + m^2\frac{v^2}{w^2}\right) - 2m \tag{B3}$$

The calculation is rather long and tedious. We will only outline the main ideas here. First, let us define

$$v_\alpha(\mathbf{k}, w) = A_\alpha(\mathbf{k}, w) + iB_\alpha(\mathbf{k}, w); \qquad \phi_\alpha(\mathbf{k}, w) = C_\alpha(\mathbf{k}, w) + iD_\alpha(\mathbf{k}, w) \tag{B4}$$

$$\hat{\rho}(\mathbf{k}, w) = E(\mathbf{k}, w) + iF(\mathbf{k}, w); \qquad \hat{\sigma}(\mathbf{k}, w) = G(\mathbf{k}, w) + iH(\mathbf{k}, w) \tag{B5}$$

Because the fields $v_\alpha(\mathbf{x}, t)$, $\phi_\alpha(\mathbf{x}, t)$, $\hat{\rho}(\mathbf{x}, t)$, and $\hat{\sigma}(\mathbf{x}, t)$ are real, $A_\alpha(\mathbf{k}, w)$, $C_\alpha(\mathbf{k}, w)$, $E(\mathbf{k}, w)$, and $G(\mathbf{k}, w)$ are even under inversion. $B_\alpha(\mathbf{k}, w)$, $D_\alpha(\mathbf{k}, w)$, $F(\mathbf{k}, w)$, and $H(\mathbf{k}, w)$ are odd under inversion. Hence, the range of integration in the Fourier space has to be restricted. In the following, the integration symbol $\int'$ reminds us that the integration is restricted to the half-space. It is easy to show that

$$\begin{aligned}
\tilde{Z}_0 \equiv \int \prod_{\alpha=1}^{d} & [\mathscr{D}A_\alpha(\mathbf{k}, w)][\mathscr{D}B_\alpha(\mathbf{k}, w)][\mathscr{D}C_\alpha(\mathbf{k}, w)][\mathscr{D}D_\alpha(\mathbf{k}, w)] \\
& \times [\mathscr{D}E(\mathbf{k}, w)][\mathscr{D}F(\mathbf{k}, w)][\mathscr{D}G(\mathbf{k}, w)][\mathscr{D}H(\mathbf{k}, w)] \\
& \times \exp\left\{\int' \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{dw}{2\pi}\mathscr{L}\right\}
\end{aligned} \tag{B6}$$

$$\begin{aligned}
\mathscr{L} \equiv {} & A_\alpha(\mathbf{k}, w)[l_\alpha(-\mathbf{k}, -w) + l_\alpha(\mathbf{k}, w)] \\
& + iB_\alpha(\mathbf{k}, w)[l_\alpha(-\mathbf{k}, -w) - l_\alpha(\mathbf{k}, w)] \\
& + C_\alpha(\mathbf{k}, w)\,\tilde{X}_\alpha(\mathbf{k}, w) + D_\alpha(\mathbf{k}, w)\,\tilde{Y}_\alpha(\mathbf{k}, w) \\
& + E(\mathbf{k}, w)\{[J(-\mathbf{k}, -w) + J(\mathbf{k}, w)] - 2ik^{2n}k_\alpha D_\alpha(\mathbf{k}, w)\} \\
& + F(\mathbf{k}, w)\{i[J(-\mathbf{k}, -w) - J(\mathbf{k}, w)] + 2ik^{2n}k_\alpha C_\alpha(\mathbf{k}, w)\} \\
& + G(\mathbf{k}, w)\,X(\mathbf{k}, w) + H(\mathbf{k}, w)\,Y(\mathbf{k}, w)
\end{aligned} \tag{B7}$$

$$\begin{aligned}
\tilde{X}_\alpha(\mathbf{k}, w) \equiv {} & ia[l_\alpha(-\mathbf{k}, -w) + l_\alpha(\mathbf{k}, w)] + [h_\alpha(-\mathbf{k}, -w) + h_\alpha(\mathbf{k}, w)] \\
& - 2ik^{2+2n}A_\alpha(\mathbf{k}, w) - 2k^{2n}\left(\frac{iw}{v}\right)B_\alpha(\mathbf{k}, w) \\
& - 2ibk^{2n}k_\alpha k_\beta A_\beta(\mathbf{k}, w)
\end{aligned} \tag{B8}$$

$$\begin{aligned}
\tilde{Y}_\alpha(\mathbf{k}, w) \equiv {} & -a[l_\alpha(-\mathbf{k}, -w) - l_\alpha(\mathbf{k}, w)] + i[h_\alpha(-\mathbf{k}, -w) - h_\alpha(\mathbf{k}, w)] \\
& - 2ik^{2+2n}B_\alpha(\mathbf{k}, w) + 2k^{2n}\left(\frac{iw}{v}\right)A_\alpha(\mathbf{k}, w) \\
& - 2ibk^{2n}k_\alpha k_\beta B_\beta(\mathbf{k}, w)
\end{aligned} \tag{B9}$$

$$X(\mathbf{k}, w) \equiv [I(-\mathbf{k}, -w) + I(\mathbf{k}, w)] - 2i\frac{w}{v}F(\mathbf{k}, w) + 2imk_\alpha B_\alpha(\mathbf{k}, w) \tag{B10}$$

$$Y(\mathbf{k}, w) \equiv i[I(-\mathbf{k}, -w) - I(\mathbf{k}, w)] + 2i\frac{w}{v}E(\mathbf{k}, w) - 2imk_\alpha A_\alpha(\mathbf{k}, w) \tag{B11}$$

Next, we integrate out $G(\mathbf{k}, w)$ and $H(\mathbf{k}, w)$. The trick is to insert a constant to generate the Gaussian integral:

$$\begin{aligned}
&\int [\mathscr{D}G(\mathbf{k}, w)] \exp\left\{\int' \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{dw}{2\pi} G(\mathbf{k}, w)\, X(\mathbf{k}, w)\right\} \\
&\quad \equiv \frac{1}{N}\int [\mathscr{D}S(\mathbf{k}, w)][\mathscr{D}G(\mathbf{k}, w)] \\
&\quad\quad \times \exp\left\{\int' \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{dw}{2\pi}\left\{-\frac{1}{2}[S(\mathbf{k}, w) - G(\mathbf{k}, w)]^2 + G(\mathbf{k}, w)\, X(\mathbf{k}, w)\right\}\right\} \\
&\quad \equiv \int [\mathscr{D}S(\mathbf{k}, w)] \exp\left\{\int' \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{dw}{2\pi}\left[\frac{1}{2}X^2(\mathbf{k}, w) + S(\mathbf{k}, w)\, X(\mathbf{k}, w)\right]\right\}
\end{aligned} \tag{B12}$$

$$N \equiv \int [\mathscr{D}S(\mathbf{k}, w)] \exp\left\{-\frac{1}{2}\int' \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{dw}{2\pi}[S(\mathbf{k}, w) - G(\mathbf{k}, w)]^2\right\} \tag{B13}$$

where we have absorbed the constant N into the definition of the functional integration measure. We repeat this procedure until we integrate out all fields in the problem. We will then obtain the result in Eq. (B1).

APPENDIX C

We have

$$\begin{aligned}
\tilde{Z}_0 \equiv &\int [\mathscr{D}\hat{\rho}][\mathscr{D}\hat{\sigma}][\mathscr{D}u_\alpha][\mathscr{D}\phi_\alpha] \exp\{\langle\!\langle u_\alpha, l_\alpha\rangle\!\rangle \\
&+ \langle\!\langle \phi_\alpha, h_\alpha\rangle\!\rangle + \langle\!\langle \hat{\rho}, J\rangle\!\rangle + \langle\!\langle \hat{\sigma}, I\rangle\!\rangle\} \\
&\times \exp\left\{-\int k^{2n}[ak^2\delta_{\alpha\beta} + ek_\alpha k_\beta]\, \phi_\alpha(\mathbf{k}, w)\, \phi_\beta(-\mathbf{k}, -w)\right\} \\
&\times \exp\left\{-i\int k^{2n}\left[\left(k^2 + \frac{iw}{v}\right)\delta_{\alpha\beta} + bk_\alpha k_\beta\right]\phi_\alpha(\mathbf{k}, w)\, u_\beta(-\mathbf{k}, -w)\right\}
\end{aligned}$$

$$\times \exp\left\{-\int [k^{2n}k_\alpha \phi_\alpha(\mathbf{k}, w)\,\hat{\rho}(-\mathbf{k}, -w) + mk_\alpha \hat{\sigma}(\mathbf{k}, w)\,u_\alpha(-\mathbf{k}, -w)]\right\}$$

$$\times \exp\left\{i\int\left[-\left(\frac{iw}{v}\right)\hat{\sigma}(\mathbf{k}, w)\,\hat{\rho}(-\mathbf{k}, -w) + fk_\alpha \hat{\sigma}(\mathbf{k}, w)\,\phi_\alpha(-\mathbf{k}, -w)\right]\right\}$$

$$= \exp\left\{\int (-i)\frac{[k^2 + iw/v + bk^2 - im(v/w)\,k^2]\,vk_\alpha}{[k^4 + w^2/v^2 + ck^2]\,w}\,l_\alpha(-\mathbf{k}, -w)\,I(\mathbf{k}, w)\right\}$$

$$\times \exp\left\{\int\left[\frac{v}{w} + (-i)\frac{[k^2 + iw/v + bk^2 - im(v/w)\,k^2]\,mv^2k^2}{[k^4 + w^2/v^2 + ck^2]\,w^2}\right]\right.$$

$$\left.\times I(\mathbf{k}, w)\,J(-\mathbf{k}, -w)\right\}$$

$$\times \exp\left\{\int a\frac{\{2[(a+e)/a]k^2 - (f/am)[k^2 + iw/v + bk^2 - im(v/w)k^2]\}\,mvk_\alpha}{k^{2n}[k^4 + w^2/v^2 + ck^2]\,w}\right.$$

$$\left.\times l_\alpha(-\mathbf{k}, -w)\,J(\mathbf{k}, w)\right\}$$

$$\times \exp\left\{\int(-i)\frac{[k^2 + iw/v + bk^2 - im(v/w)k^2]mvk_\alpha}{k^{2n}[k^4 + w^2/v^2 + ck^2]\,w}\,h_\alpha(\mathbf{k}, w)J(-\mathbf{k}, -w)\right\}$$

$$\times \exp\left\{\int i\frac{[c(k^2 + iw/v) + (k^4 + w^2/v^2)(-b + im(v/w))]\,k_\alpha k_\beta}{k^{2n}[k^4 + w^2/v^2][k^4 + w^2/v^2 + ck^2]}\right.$$

$$\left.\times h_\alpha(\mathbf{k}, w)\,l_\beta(-\mathbf{k}, -w)\right\}$$

$$\times \exp\left\{\int(-a)\frac{[ck^2 - (e/a)(k^4 + w^2/v^2)]k_\alpha k_\beta}{k^{2n}[k^4 + w^2/v^2][k^4 + w^2/v^2 + ck^2]}\,l_\alpha(\mathbf{k}, w)\,l_\beta(-\mathbf{k}, -w)\right\}$$

$$\times \exp\left\{\int\left[\frac{(-i)\,h_\alpha(\mathbf{k}, w)\,l_\alpha(-\mathbf{k}, -w)}{k^{2n}(k^2 - iw/v)} + a\frac{l_\alpha(\mathbf{k}, w)\,l_\alpha(-\mathbf{k}, -w)}{k^{2n-2}(k^4 + w^2/v^2)}\right]\right\}$$

$$\times \exp\left\{\int\frac{[m(a+e) - f(1+b)]\,mv^2k^4}{k^{2n}[k^4 + w^2/v^2 + ck^2]\,w^2}\,J(-\mathbf{k}, -w)\,J(\mathbf{k}, w)\right\} \tag{C1}$$

where

$$\langle\langle u_\alpha, l_\alpha\rangle\rangle \equiv \int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{dw}{2\pi}\,l_\alpha(-\mathbf{k}, -w)\,u_\alpha(\mathbf{k}, w) \tag{C2}$$

$$\int \equiv \int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{dw}{2\pi} \tag{C3}$$

$$c \equiv k^2\left(b^2 + 2b + m^2\frac{v^2}{w^2}\right) - 2m \tag{C4}$$

We will not present the derivation of Eq. (C1) here. It is a standard Gaussian integral. $u_\alpha(\mathbf{x}, t)$, $\phi_\alpha(\mathbf{x}, t)$, $\hat{\rho}(\mathbf{x}, t)$, and $\hat{\sigma}(\mathbf{x}, t)$ are real fields.

In Eqs. (C1) and (B1), there is some ambiguity in the w integration, because some propagators have poles in the real w axis. We will put in some small imaginary part to avoid this singularity. The sign for this small imaginary part is chosen to be consistent with the hypothetical case where $\hat{\rho}$ has a small diffusive part. See Eq. (3.36).

The motivation for the above prescription can be understood as follows. In Eq. (2.10), the linear part of the equation for $\hat{\rho}$ contains only $\partial\hat{\rho}/\partial t$. Since w has a dimension of Λ^2, it is likely that diverging counterterm of the form $k^2\hat{\rho}$ will be generated. However, the equation of continuity (from the mass conservation law) has no such diffusive part. In order to ensure the renormalizability of the theory, we have introduced "extra operators" in Eq. (3.10). Indeed, $e_2\partial_\alpha$NS will provide such a diffusive part for $\hat{\rho}$ in the counterterm action. Hence, our prescription in avoiding the singularity in the w integration can be regarded as a limiting case of a larger theory where the percentage density fluctuation, i.e., $\hat{\rho}$, has a diffusive part in the renormalized equation of motion.

APPENDIX D

In this Appendix, I list all the integrals appearing in the calculation of the self-energy and the vertex corrections, at the one-loop level. I use *dimensional regularization* and I write only the leading pole term. From Eq. (D1) to Eq. (D7), $\varepsilon = 4 - d$. From Eq. (D8) to Eq. (D13), $\varepsilon = 6 - d$.

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{(k_\alpha + q_\alpha)(k_\beta + q_\beta)}{(\mathbf{k}+\mathbf{q})^2\,[k^2 + (\mathbf{k}+\mathbf{q})^2 + iw/v]}$$

$$= \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{k_\alpha k_\beta}{k^2[k^2 + (\mathbf{k}-\mathbf{q})^2 + iw/v]}$$

$$\approx \frac{1}{(4\pi)^2}\cdot\frac{1}{\varepsilon}\cdot\left[-\frac{1}{12}q^2\delta_{\alpha\beta} - \frac{1}{8}\frac{iw}{v}\delta_{\alpha\beta} + \frac{1}{12}q_\alpha q_\beta\right] \tag{D1}$$

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{k_\alpha(k_\beta + q_\beta)}{k^2[k^2 + (\mathbf{k}+\mathbf{q})^2 + iw/v]}$$

$$= \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{(k_\alpha - q_\alpha)\,k_\beta}{(\mathbf{k}-\mathbf{q})^2\,[k^2 + (\mathbf{k}-\mathbf{q})^2 + iw/v]}$$

$$\approx \frac{1}{(4\pi)^2}\cdot\frac{1}{\varepsilon}\cdot\left[-\frac{1}{12}q^2\delta_{\alpha\beta} - \frac{1}{8}\frac{iw}{v}\delta_{\alpha\beta} - \frac{1}{6}q_\alpha q_\beta\right] \tag{D2}$$

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{(k_\alpha + q_\alpha)}{(\mathbf{k}+\mathbf{q})^2\,[k^2 + (\mathbf{k}+\mathbf{q})^2 + iw/v}$$

$$= \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{k_\alpha}{k^2[k^2 + (\mathbf{k}-\mathbf{q})^2 + iw/v]}$$

$$\approx \frac{1}{(4\pi)^2} \cdot \frac{1}{\varepsilon} \cdot \left[\frac{1}{4} q_\alpha\right] \quad \text{(D3)}$$

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{(k_u + q_u)}{k^2[k^2 + (\mathbf{k}+\mathbf{q})^2 + iw/v]}$$

$$= \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{k_u}{(\mathbf{k}-\mathbf{q})^2\,[k^2 + (\mathbf{k}-\mathbf{q})^2 + iw/v]}$$

$$\approx \frac{1}{(4\pi)^2} \cdot \frac{1}{\varepsilon} \cdot \left[\frac{3}{4} q_u\right] \quad \text{(D4)}$$

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{k_\alpha(k_\beta + q_\beta)}{k^2(\mathbf{k}+\mathbf{q})^2\,[k^2 + (\mathbf{k}+\mathbf{q})^2 + iw/v]}$$

$$\approx \frac{1}{(4\pi)^2} \cdot \frac{1}{\varepsilon} \cdot \left[\frac{1}{4} \delta_{\alpha\beta}\right] \quad \text{(D5)}$$

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{k^2[k^2 + (\mathbf{k}+\mathbf{q})^2 + iw/v]}$$

$$= \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{(\mathbf{k}-\mathbf{q})^2\,[k^2 + (\mathbf{k}-\mathbf{q})^2 + iw/v]}$$

$$\approx \frac{1}{(4\pi)^2} \cdot \frac{1}{\varepsilon} \cdot [1] \quad \text{(D6)}$$

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{k_a k_b}{(\mathbf{k}+\mathbf{t})^2\,[k^2 + (\mathbf{k}+\mathbf{u})^2 + iE/v][k^2 + (\mathbf{k}+\mathbf{v})^2 + iF/v]}$$

$$\approx \frac{1}{(4\pi)^2} \cdot \frac{1}{\varepsilon} \cdot \left[\frac{1}{8} \delta_{ab}\right] \quad \text{(D7)}$$

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{(k_\alpha + q_\alpha)(k_\beta + q_\beta)}{(\mathbf{k}+\mathbf{q})^4\,[k^2 + (\mathbf{k}+\mathbf{q})^2 + iw/v]}$$

$$= \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{k_\alpha k_\beta}{k^4[k^2 + (\mathbf{k}-\mathbf{q})^2 + iw/v]}$$

$$\approx \frac{1}{(4\pi)^3} \cdot \frac{1}{\varepsilon} \cdot \left[-\frac{1}{32} q^2 \delta_{\alpha\beta} - \frac{1}{24} \frac{iw}{v} \delta_{\alpha\beta} + \frac{1}{48} q_\alpha q_\beta\right] \quad \text{(D8)}$$

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{k_\alpha(k_\beta+q_\beta)}{k^4[k^2+(\mathbf{k}+\mathbf{q})^2+iw/v]}$$

$$= \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{(k_\alpha-q_\alpha)\,k_\beta}{(\mathbf{k}-\mathbf{q})^4\,[k^2+(\mathbf{k}-\mathbf{q})^2+iw/v]}$$

$$\approx \frac{1}{(4\pi)^3}\cdot\frac{1}{\varepsilon}\cdot\left[-\frac{1}{32}q^2\delta_{\alpha\beta}-\frac{1}{24}\frac{iw}{v}\delta_{\alpha\beta}-\frac{1}{16}q_\alpha q_\beta\right] \tag{D9}$$

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{(k_\alpha+q_\alpha)}{(\mathbf{k}+\mathbf{q})^4\,[k^2+(\mathbf{k}+\mathbf{q})^2+iw/v]}$$

$$= \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{k_\alpha}{k^4[k^2+(\mathbf{k}-\mathbf{q})^2+iw/v]}$$

$$\approx \frac{1}{(4\pi)^3}\cdot\frac{1}{\varepsilon}\cdot\left[\frac{1}{12}q_\alpha\right] \tag{D10}$$

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{(k_u+q_u)}{k^4[k^2+(\mathbf{k}+\mathbf{q})^2+iw/v]}$$

$$= \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{k_u}{(\mathbf{k}-\mathbf{q})^4\,[k^2+(\mathbf{k}-\mathbf{q})^2+iw/v]}$$

$$\approx \frac{1}{(4\pi)^3}\cdot\frac{1}{\varepsilon}\cdot\left[\frac{5}{12}q_u\right] \tag{D11}$$

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{k_a k_b}{(\mathbf{k}+\mathbf{t})^4\,[k^2+(\mathbf{k}+\mathbf{u})^2+iE/v][k^2+(\mathbf{k}+\mathbf{v})^2+iF/v]}$$

$$\approx \frac{1}{(4\pi)^3}\cdot\frac{1}{\varepsilon}\cdot\left[\frac{1}{24}\delta_{ab}\right] \tag{D12}$$

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